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LETTER TO THE EDITOR

Directed percolation: pseudo-correlation length

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Abstract. We show that the perpendicular correlation length $\xi_\perp$ in directed percolation does not behave as a length under renormalisation but is a length times an angle. This has consequences for the renormalisation group analysis and hyperscaling which are discussed.

Percolation (for introductions to the subject see review articles by Stauffer (1979) and Essam (1980), and in particular directed percolation, has been the subjects of much current interest.

The interest in directed percolation is stimulated by the fact that it is in a different universality class than ordinary percolation (Obukhov 1980, Redner 1981, Redner and Brown 1981, Reynolds 1981) and by its similarity to Reggeon field theory (Cardy and Sugar 1980) and Markov process that occur in chemistry and biology (Grassberger and Torre 1979, Schlögl 1972).

Directed percolation is defined as follows (Broadbent and Hammersley 1957): consider a square lattice on which horizontal bonds are present with probability p_H and vertical bonds with a probability p_V. We further restrict the problem so that bonds which are present have a direction: down for vertical bonds and to the left for horizontal bonds (see figure 1).

A correlation length $\xi_\parallel(\varphi)$ can be defined by writing the probability of a site at the origin being connected to a site at a position given by $\boldsymbol{R}$ as $(P(\boldsymbol{R}, p) = \exp[-|\boldsymbol{R}|/\xi_\parallel(\varphi)]$ in the limit $|\boldsymbol{R}| \to \infty$. As the percolation threshold is approached

$$\xi_\parallel(\varphi) \sim [p - p_c(\theta)]^{-}\nu_\parallel(\varphi). \tag{1}$$

As opposed to ordinary percolation $p_c(\theta)$ and $\nu_\parallel(\theta)$ depend on the angle the vector $\boldsymbol{R}$ makes with the bond directions. For example, if we consider the case $p_H = p_V = p$ and take the angle θ to be measured from the diagonal of the square face (see figure 2) it was shown by Domany and Kinzel (1981) that $\nu_\parallel(\varphi = 0) \neq \nu_\parallel(\varphi \neq 0)$.

Another feature of directed percolation that distinguishes it from ordinary percolation is the apparent divergence of two lengths. In addition to the length defined above one can ask for the probability that two sites separated by a distance ρ along a direction *perpendicular* to $\boldsymbol{R}$ belong to the same cluster (see figure 3). This defines a correlation length $\xi_\perp$ which was shown by Kinzel and Yeomans (1981) to diverge as

$$\xi_\perp \sim [p - p_c(\varphi)]^{-}\nu_\perp(\varphi) \tag{2}$$

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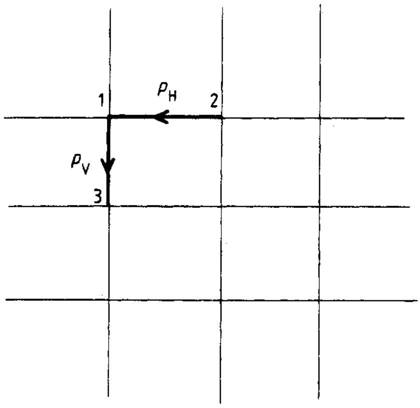

Figure 1. A cluster of 3 sites in the directed percolation problem. There is a path from site 2 to site 3 but no path from site 3 to site 2.

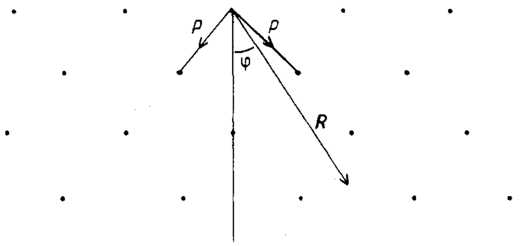

Figure 2. The arrows labelled p are along the direction of the face of the square. Percolation is observed along the $\boldsymbol{R}$ direction that makes an angle φ with the diagonal of the face.

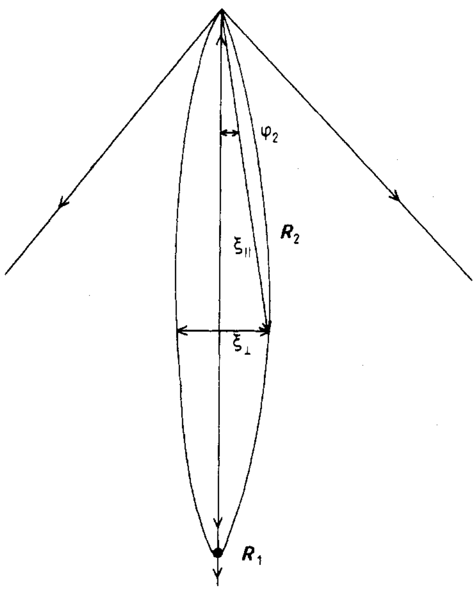

Figure 3. The direction for $\boldsymbol{R}_1$ is taken along $\varphi_1 = 0$. The vector $\boldsymbol{R}_2$ makes an angle φ_2 with $\boldsymbol{R}_1$. $\xi_{\parallel}$ and $\xi_{\perp}$ are the longitudinal and transverse diameters of the percolating cone.

where $\nu_\perp(\varphi) \neq \nu_\parallel(\varphi)$ and as before $\nu_\perp(\theta=0) \neq \nu_\perp(\varphi \neq 0)$. This length $\xi_\perp$ gives the width of the cigar-shaped percolating cone (figure 3). In order to understand this phenomena Kinzel and Yoemans introduced an anisotropy exponent θ defined by the relation

$$\theta = \nu_\parallel / \nu_\perp. \tag{3}$$

Equations (1) to (3) and the scaling relation

$$\xi'_\parallel = \xi_\parallel / b \tag{4}$$

lead to the result

$$\xi'_\perp = \xi_{\perp/b} 1/\theta. \tag{5}$$

Finite size renormalisation group (RG) calculations give (Kinzel and Yeomans 1981) for $\varphi = 0$

$$\theta = 1.582 \pm 0.001 \qquad \nu_\perp = 1.098 \pm 0.005. \tag{6}$$

In this letter we present arguments that relate the anisotropy exponent θ of Kinzel and Yeomans to the crossover exponent x, defined by

$$[p - p_c(\varphi)] = (\varphi - \varphi_c)^{1/x} \tag{7}$$

where p and φ are varied in such a way as to remain on the critical curve. The argument proceeds as follows.

In order to calculate the exponent $\nu_\perp(\varphi)$ we must have a more precise definition of what it means to have two sites belonging to the same cluster if they are separated by a distance ρ along the perpendicular to $\boldsymbol{R}$ (figure 3). Consideration of figure 3 immediately yields the conclusion that there is no connected path of directed bonds that links two sites perpendicular to $\boldsymbol{R}$. Two such sites can only be said to belong to the same cluster if they are both connected to the same site along two directions $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ (figure 3). We shall call such a common point a reference point.

The transverse correlation length is then defined as the distance measured from site 1 (which is along $\boldsymbol{R}_1$, a distance $|\boldsymbol{R}_1|$ from the reference point) to site 2 (along $\boldsymbol{R}_2$, $|\boldsymbol{R}_3|$ from the reference point) such that $\boldsymbol{R}_1 - \boldsymbol{R}_2$ is perpendicular to $\boldsymbol{R}_1$ and sites 1 and 2 are connected to the reference site, i.e. $|\boldsymbol{R}_1|$, $|\boldsymbol{R}_2| \approx \xi_\parallel$.

It now follows that

$$\xi_\perp = |\boldsymbol{R}_1 - \boldsymbol{R}_2| = |\boldsymbol{R}_1| \tan(\varphi_1 - \varphi_2) \tag{8}$$

As p approaches $p_c(\varphi_1)$ we must have that $\varphi_1 \to \varphi_2$ so that $\xi_\parallel(\varphi_2)$ also diverges and sites 1 and 2 remain in the same cluster. Defining $\varphi_1 - \varphi_2 = \Delta\varphi$ this requirement implies that

$$\xi_\perp = \xi_\parallel \Delta\varphi \tag{9}$$

as $p \to p_c(\varphi_1)$.

From equation (9) it is clear that although $\xi_\perp$ has the units of a length (it is in fact an arc length), it will not behave under RG as a length since $\Delta\varphi$ is also renormalised. This will be seen in more detail below.

We shall refer to this length $\xi_\perp$ as a pseudolength.

Equation (9) will enable us to derive various relations for θ. Under RG we must have that

$$\xi'_\perp / \xi'_\parallel = b^{y_\varphi} \Delta\varphi \tag{10}$$

where b is the rescaling length of our transformation and y_φ the scaling power associated with $\Delta\varphi$. In the following we shall omit the arguments, but it is understood that all quantities may have an angle dependence. From equations (4), (5), (9) and (10) we have

$$1/\theta = 1 - y\varphi \tag{11}$$

or

$$\xi'_\perp = (1/b^{1-y\varphi})\xi_\perp. \tag{12}$$

For small values of $\Delta p = p - p_c(\varphi)$ we have $\xi'_\perp(b^{yp}\Delta p) = (1/b^{1-y\varphi})\xi_\perp(\Delta p)$. Standard scaling arguments and equation (2) give

$$\nu_\perp = (1 - y\varphi)/yp \tag{13}$$

which can be simply rewritten as

$$\nu_\perp = \nu_\parallel - x \tag{14}$$

where x is the crossover exponent defined in equation (7).

From equation (3) we can also write

$$\theta = (\nu_\parallel/(\nu_\parallel - x) = (x + \nu_\perp)/\nu_\perp. \tag{15}$$

Another important consequence of equation (9) is a more complicated form of hyperscaling than is found in standard percolation. This can be seen most easily from the requirement that hyperscaling hold, written in the following way:

$$n(\Delta p)\xi^d = \text{constant}. \tag{16}$$

For standard percolation $n(\Delta p)$ is the mean number of clusters per reference site or the percolation 'free energy per site.' Physically equation (16) is a requirement that the 'free energy' per site associated with a region the size of an incipient infinite cluster scales with the volume. In directed percolation however the volume of an incipient infinite cluster is given by $\xi_\parallel(\xi_\parallel\Delta\varphi)^{d-1}$ as $p \to p_c$. This implies that for directed percolation d should be replaced in equation (16) by $d - y_\varphi(d-1)$. This substitution gives the hyperscaling equality

$$2 - \alpha = [d - y_\varphi(d-1)]\nu_\parallel. \tag{17}$$

From equations (3) and (11) we can also write

$$2 - \alpha = [(d-1) + \theta]\nu_\perp. \tag{18}$$

The interpretation of equations (17) and (18) is that hyperscaling laws which relate correlation function exponents such as ν and (as we will see below) η (which can be chosen in the parallel or perpendicular direction) to global quantities such as α, β and γ require that $\nu_\parallel$ be associated with $d_\parallel = d - y_\varphi(d-1)$ and $\nu_\perp$ be associated with $d_\perp = d - 1 + \theta$.

It is a simple matter to see that both equations (17) and (18) reduce† to

$$2 - \alpha = \nu_\perp + \nu_\parallel \tag{19}$$

in $d = 2$.

We now derive, using arguments similar to those used above, a relation between

$$\omega_\parallel = d_\parallel - 2 + \eta_\parallel \tag{20}$$

† Similar scaling relations can be derived from mean-field considerations (S Redner, unpublished).

and

$$\omega_\perp = d_\perp - 2 + \eta_\perp.$$

At p_c the probability that two sites, one at the origin and another at $\boldsymbol{R}_1$ (labelled S_1), belong to the same cluster is given by

$$P(\boldsymbol{R}_1, p_c) = 1/|\boldsymbol{R}_1|^{d_\parallel - 2 + \eta_\parallel} = 1/|\boldsymbol{R}_1|^{\omega_\parallel}. \tag{21}$$

The probability that a site at $\boldsymbol{R}_1$ is in the same cluster as a site at a distance ρ along a direction perpendicular to $\boldsymbol{R}_1$ (labelled s_2) is given by (for $\boldsymbol{R}_1$ going to infinity)

$$P(\rho, p_c) = 1/\rho^{d_\perp - 2 + \eta_\perp} = 1/\rho^{\omega_\perp}. \tag{22}$$

From the same arguments as above we must have that

$$1/|\boldsymbol{R}_2|^{\omega_\parallel} = 1/\rho^{\omega_\perp} \tag{23}$$

and

$$\rho/|\boldsymbol{R}_2| = \Delta\varphi \tag{24}$$

where R_2 is the position vector of S_2. Equation (24) implies that ρ is again a pseudo-length. Scaling under RG then requires that

$$\omega_\parallel/\omega_\perp = 1 - y_\varphi = 1/\theta. \tag{25}$$

We illustrate these points for the special case $p_H = 1$, $p_V = p$, a model which was solved exactly for p_c and $\nu_\parallel$ (Domany and Kinzel 1981).

Domany and Kinzel have shown that the probability that two sites are connected along a vector $\boldsymbol{R}_1$ making an angle φ with the diagonal is given by

$$P(p, \alpha) = \int_{-[R_1/n_c]^{1/2}}^{\varepsilon[R_1/n_c]^{1/2}} \exp(-\tfrac{1}{2}y^2)\,\mathrm{d}y \tag{26}$$

where $\alpha = \tan\varphi$, $\varepsilon = (x - x_c)/x_c$, $\mathrm{n} = \alpha\mathrm{p}$ and $n_c = \alpha/(1+\alpha)$. From the definitions for n_c and n is is clear that the crossover exponent $x = 1$. It can be easily seen from equation (26) that $\nu_\parallel = 2$. We have immediately from the above discussion that

$$\nu_\perp = 1. \tag{27}$$

It can also be shown from equation (26) that $\omega_\parallel = 0$. From equation (25) $\omega_\perp = 0$. These values also lead to $d_\parallel = \frac{3}{2}$ and $d_\perp = 3$.

With these results and standard scaling arguments one can easily obtain

$$\beta = 0, \qquad \gamma = 3, \qquad \eta_\parallel = \tfrac{1}{2}, \qquad \eta_\perp = -1. \tag{28}$$

The value $\beta = 0$ is interesting in that it may indicate a type of first-order character for the percolation transition usually associated with one-dimensional problems.

The physical interpretation of $\beta = 0$ is the following. If we consider an angle φ for which $p_c(\varphi)$ is not the minimum possible p_c as a function of φ, then when percolation occurs at an angle φ there is an infinite cluster of finite measure that already exists on the lattice. As p then approaches $p_c(\varphi)$ from above, the measure of the infinite cluster does not go to zero as in ordinary percolation. For the special case $p_H = 1$ this is true for all angles except the special direction parallel to the p_H axis. Equation (26) can also be used as the basis for an RG calculation. Consider a cell in the two-dimensional square lattice with sides of length L (figure 4). Equation (26) allows us to calculate the probability that there exists a path across the cell at an angle φ.

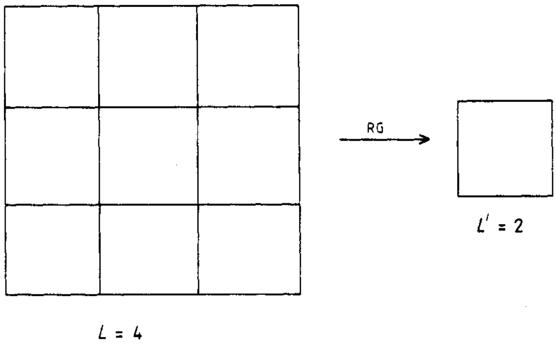

Figure 4. Renormalisation group cells for $L = 4$ and $b = 2$.

If we renormalise to a cell of length $L' = L/b$ and demand that the probability of there being a path across the cell at φ be conserved, equation (26) yields

$$\int_{-[R_1/n_c b]^{1/2}}^{\varepsilon'[R_1/n_c b]^{1/2}} \exp(-\tfrac{1}{2}y^2)\,\mathrm{d}y = \int_{-[R_1/n_c]^{1/2}}^{\varepsilon[R_1/n_c]^{1/2}} \exp(-\tfrac{1}{2}y^2)\,\mathrm{d}y. \tag{27}$$

Equation (27) immediately gives $\partial\varepsilon'/\partial\varepsilon|_{\varepsilon=0} = b^{1/2}$. From the definition of ε it is simple to see that

$$(p' - p_c) = b^{1/2}(p - p_c) \tag{28}$$

or $y_p^{-1} = \nu_\parallel = 2$.

We can also fix $p = p_c(\varphi)$ and also demand that the probability of a path across the cell at $\varphi + \Delta\varphi$ be conserved. This leads to the RG transformation

$$\int_{-[R_1/(n_c+\Delta n_c')b]^{1/2}}^{\varepsilon'[R_1/(n_c+\Delta n_c')b]^{1/2}} \exp(-\tfrac{1}{2}y^2)\,\mathrm{d}y = \int_{-[R_1/(n_c+\Delta n_c)]^{1/2}}^{\varepsilon[R_1/(n_c+\Delta n_c)]^{1/2}} \exp(-\tfrac{1}{2}y^2)\,\mathrm{d}y. \tag{29}$$

Equation (29) gives (in the limit $r_1 \to \infty$, which is its range of validity (Domany and Kinzel 1981) $\partial(\Delta n)/\partial(\Delta n') = 1$.

We can see then that the scaling fields are ε and Δn_c with eigenvalues $b^{1/2}$ and 1 respectively.

From the form of ε it is clear that $x = 1$ and $y_\varphi = \frac{1}{2}$. Clearly from the above considerations and Kinzel and Yeomans (1981) we can also predict the crossover exponent for non-trivial (e.g. $p_H = p_V = p$, $\varphi = 0$) direction.

$$x = \nu_\parallel - \nu_\perp = \nu_\parallel(1 - 1/\theta) \qquad x = 0.64 \pm 0.01.$$

In conclusion we have shown that the perpendicular correlation length is not a true length in a RG sense but is a pseudo length (i.e. a length times an angle). This has important consequences for the notion of hyperscaling and what is meant by dimension. These ideas may also have some importance for other problems with anisotropies such as Lifshitz points and anisotropic percolation. These points are being investigated.

We should like to acknowledge very interesting and useful discussions with K Binder and S Redner.

References

Broadbend S R and Hammersley 1957 *Proc. Camb. Phil. Soc.* **53** 629
Cardy J L and Sugar R L 1980 *J. Phys. A: Math. Gen.* **13** L423
Domany E and Kinzel W 1981 *Phys. Rev. Lett.* **47** 5
Essam J W 1980 *Rep. Prog. Phys.* **43** 833
Grassberger P and de la Torre A 1979 *Ann. Phys., NY* **122** 373
Kinzel W and Yoemans J M 1981 *J. Phys. A: Math. Gen.* **14** L163
Obukhov S P 1980 *Physica* A **101** 145
Redner S preprint
Redner S and Brown A C 1981 *J. Phys. A: Math. Gen.* **14** L285
Reynolds P J unpublished
Schlögl F 1972 *Z. Phys.* **253** 147
Stauffer D 1979 *Phys. Rep.* **54** 1